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Incentives for Strategic Behavior in Fisher Market Games

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Abstract

In a Fisher market game, a market equilibrium is computed in terms of the utility functions and money endowments that agents reported. As a consequence, an individual buyer may misreport his private information to obtain a utility gain. We investigate the extent to which an agent's utility can be increased by unilateral strategic plays and prove that the percentage of this improvement is at most 2 for markets with weak gross substitute utilities. Equivalently, we show that truthfully reporting is a 0.5-approximate Nash equilibrium in this game. To identify sufficient conditions for truthfully reporting being close to Nash equilibrium, we conduct a parameterized study on strategic behaviors and further show that the ratio of utility gain decreases linearly as buyer's initial endowment increases or his maximum share of an item decreases. Finally, we consider collusive behavior of a coalition and prove that the utility gain is bounded by $1/(1-$ maximum share of the collusion). Our findings justify the truthful reporting assumption in Fisher markets by a quantitative study on participants incentive, and imply that under large market assumption, the utility gain of a buyer from manipulations diminishes to 0.

Introduction

The Internet and World Wide Web have created a possibility for buyers and sellers to meet at a marketplace in which pricing and allocations can be determined more efficiently and effectively than ever before. Market equilibrium, a vital notion in classic economic theory, ensures optimum fairness and efficiency and has become a paradigm for practical applications in computer science. Understanding its properties and computation has been one of the central questions in economics and computer science. In this paper, we consider the Fisher market model (Brainard and Scarf 2000), in which a market maker sells divisible items of unit supply each to potential buyers, each endowed with an initially amount of cash and a utility function. At a market equilibrium, all products are sold out, all cash is spent, and, most importantly, the set of items purchased by each buyer maximizes his utility for the given equilibrium prices constrained by his initial endowment. It has been shown that a market equilibrium always exists given mild assumptions (Arrow and Debreu 1954).

However, there is a major issue on the application of market equilibrium in that it has not taken strategic behavior of the participants into consideration: In a Fisher market, market equilibrium prices and associated allocations, computed in terms of utility functions and money endowments, may change even if one participant has a change in its utility function or endowment. Hence, one may misreport his private information if it results in a favorable solution.

Example 1. *Consider a market with two items and two buyers with endowments* $(\epsilon, 1-\epsilon)$ *and utility functions* $u_1(x,y) = \epsilon x + (1-\epsilon)y, u_2(x,y) = \epsilon^2 x + (1-\epsilon^2)y$*, respectively. When both buyers report truthfully, the equilibrium price is* $\mathbf{p} = (\epsilon, 1-\epsilon)$ *and the first player gets all fraction of the first item with utility* ϵ*. If the first buyer strategically reports* $u_1' = u_2$*, then the equilibrium price is* $\mathbf{p}' = (\epsilon^2, 1-\epsilon^2)$ *and he gets the bundle with* $x = 1, y = \epsilon/(1+\epsilon)$ *for which his utility is* $\epsilon(1 + (1-\epsilon)/(1+\epsilon))$*. When* $\epsilon = 1/2$*, the first player improves his utility from* $1/2$ *to* $5/6$*. When* ϵ *goes to* 0*, he doubles his utility from* ϵ *to a value close to* 2ϵ*.*

Based on this observation, Adsul et al. (2010) formulated the *Fisher market game* and studied the strategic behaviors by examining the induced Nash equilibrium. Nonetheless, the investigation on Nash equilibrium suffers from two major drawbacks. Nash equilibrium is shown to be computationally intractable (Daskalakis, Goldberg, and Papadimitriou 2009), even for two-player games (Chen, Deng, and Teng 2009) or within a small constant approximation factor (Rubinstein 2015). Even if equilibria can be computed efficiently, a recent work by Brânzei et al. (2014) shows that their social welfare might be far from the one in market under truthfully reporting.

In this paper, we quantitatively measure a buyer's utility gain by strategic plays by adopting the notion of *incentive ratio* (Chen, Deng, and Zhang 2011). Incentive ratio is defined as the factor of the largest possible utility gain that a participant can achieve by behaving strategically, given that all other participants have their strategies unchanged. Incentive ratio characterizes the extent to which utilities can be increased by strategic manipulations. A smaller incentive ratio implies that a buyer has less incentive to influence the market price formation through (complicated) strategic considerations by a significant effort to collect all utility functions of other market participants.

First of all, we consider a widely studied class of *weak* *gross substitute* (WGS) preferences. Informally, under WGS preferences, when the price of an item increases, a buyer will not decrease his consumption on items whose prices do not change. We show that incentive ratio is at most 2 if all buyers have WGS utility functions. Prior to our results, constant bounds on incentive ratio are only known for markets with three specific utilities, Leontief (Chen, Deng, and Zhang 2011), Linear and Cobb-Douglas (Chen et al. 2012). Although the latter two functions satisfy WGS, our proof techniques are different from and much simpler than theirs. Another interpretation of our result is that, truthfully reporting turns out to be a 0.5-approximate Nash equilibrium in Fisher market game. Therefore, no one would manipulate to gain twice as much utility. In complement to the above upper bounds, we illustrate a simple example where a buyer's incentive ratio could be unbounded if the utility function is additive piecewise linear, which does not satisfy WGS. Our approach escapes from the above two curses on studying exact Nash equilibrium, i.e. computational intractability and economic inefficiency, by considering its approximate alternative. This relaxed version of Nash equilibrium has been extensively studied in game theory and artificial intelligence (Daskalakis, Mehta, and Papadimitriou 2007; Ponsen, De Jong, and Lanctot 2011; Fearnley et al. 2013; Czumaj, Fasoulakis, and Jurdziński 2015).

Nevertheless, an ϵ-approximate Nash equilibrium is hardly a compelling proposal unless ϵ is very small, because each buyer knows how to make an attractive profit by deviating from it. To overcome this obstacle, we conduct a parameterized study on incentive ratio to understand individual strategic behaviors and identify sufficient conditions for markets where buyers have little incentive to deviate from truthfully reporting. We choose two important parameters that reflects the influence of an individual buyer on the market. One is the initial normalized endowment of a buyer i denoted by e_i, and the other one is α_i, his maximum share, which is the largest equilibrium allocation among all individual items (each a unit in total). We show that the incentive ratio of the buyer is at most $2-e_i$ and $1+\alpha_i$, respectively, if the utility functions satisfy WGS and a natural homogeneous condition that is extensively used in economics (Eisenberg 1961). Intuitively, if a buyer dominates a particular item, then he has considerable market power and may influence market prices; and, conversely, the more endowment a buyer has, the less incentive he may have to manipulate, since there is less value left in the market to be gained by his manipulative behavior.

Finally, we consider coordinated decisions in a market taken by a group of buyers to influence market prices to improve their utilities. We again make a parameterized study on incentive ratio to gauge a coalition's incentive for collusive behavior, which is defined as the maximum factor of utility gain from the members of a collusion, given that no one in the collusion is worse off. While in general the incentive ratio of a collusion S can be unbounded in the worst case, we show that the incentive ratio is at most $1/(1-\alpha_S)$ for homogeneous utilities that satisfy the weak gross substitute condition, where α_S is the maximum share that S obtains on any individual item in equilibrium allocations.

Our parameterized study on incentive ratio directly implies that strategic manipulations of an individual buyer is of little impact on a market if his share is small. In particular, for a large market with replicated economies, the maximum share of any buyer diminishes to 0 as the market grows, and thus, the incentive ratio converges to 1. This gives a quantitative reinterpretation of the classical economic statement that incentives for strategy behavior in market equilibrium mechanism decreases as the market grows (Roberts and Postlewaite 1976; Otani and Sicilian 1982; Jackson and Manelli 1997). Furthermore, our results can be also adopted in finite markets and carried over to the scenarios that allow collusive strategic behavior, that seems to represent the most vulnerable point towards justifying the assumption of price-taking (i.e. truthful) behavior as pointed out by Johansen (1977). Similar relations between large game assumption and strategy behavior have been also established in stable matching (Immorlica and Mahdian 2005), λ-continuous and anonymous games (Gradwohl and Reingold 2008), market design (Azevedo and Budish 2012), envy-free pricing (Anshelevich, Kar, and Sekar 2015) and bandwidth allocation (Cheng et al. 2015).

Preliminaries

In a given Fisher market, there are n buyers and m divisible goods (items, interchangeably) of unit quantity each. We use $[n] = \{1, 2, \ldots, n\}$ and $[m] = \{1, 2, \ldots, m\}$ to denote the set of buyers and items, respectively. Each buyer has an initial cash endowment e_i, which is normalized [1] to be $\sum_{i\in[n]} e_i = 1$, and a utility function $u_i : [0,1]^m \to \mathbb{R}$. That is, for a given allocation $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{im}) \in [0,1]^m$, where x_{ij} is the amount that buyer i receives from item j, $u_i(\mathbf{x}_i)$ denotes the obtained utility of buyer i. We assume that utility functions are monotone (i.e., $u(\mathbf{x}) \geq u(\mathbf{x}')$, for any $\mathbf{x} \geq \mathbf{x}'$) and normalized to be 0 when the allocation is empty (i.e., $u(\mathbf{0}) = 0$).

An outcome of the market is represented by a tuple $(\mathbf{p}, \mathbf{x})$, where $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ is a price vector of all items and $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ is an allocation vector of all buyers. We say that $\mathbf{x}_i$ is an optimal allocation for buyer i with respect to a price vector $\mathbf{p}$ if $\mathbf{x}_i$ maximizes his utility function $u_i(\mathbf{y})$ subject to the endowment constraint $\mathbf{p} \cdot \mathbf{y} \leq e_i$. An outcome is called a *market equilibrium* if the following two conditions hold.

- Market clearance: All items are sold out and all cash endowments are spent, that is $\sum_{i\in[n]} x_{ij} = 1$ for all item j, and $\sum_{j\in[m]} p_j = \sum_{i\in[n]} e_i = 1$.
- Individual optimality: The market allocation $\mathbf{x}_i$ is an optimal allocation for each buyer i with respect to the price vector $\mathbf{p}$.

Utility Functions

We review some standard definitions of utility functions.

[1] In some of the examples described in the paper endowments may not be normalized for the ease of presentation.

Definition 1 (Concavity)**.** A utility function $u(\cdot)$ is said to be *concave* if for any $\mathbf{x}, \mathbf{x}' \in [0,1]^m$ and for any $t \in [0,1]$,

$$u\big(t \cdot \mathbf{x} + (1-t) \cdot \mathbf{x}'\big) \geq t \cdot u(\mathbf{x}) + (1-t) \cdot u(\mathbf{x}')$$

Arrow and Debreu (1954) established that a market equilibrium always exists under the assumption that utility functions are concave. In the following discussions, all utility functions are assumed to be concave.

Definition 2 (Demand set)**.** For a given price vector $\mathbf{p}$, the demand set of buyer i, denoted by $D_i(\mathbf{p}, e_i)$, is the set of all optimal allocations of the following program:

$$\max \quad u_i(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{p} \cdot \mathbf{x} \leq e_i \tag{1}$$

A very nice characterization of the demand set $D_i(\mathbf{p}, e_i)$ says that the marginal bang-per-buck ratio of all purchased items (which measures the per-buck utility of the items) is the same in an optimal allocation. The following proposition relates different allocations to their respective utilities.

Proposition 2. *Let $\mathbf{y}_i = (y_{ij})_j \in D_i(\mathbf{p}, e_i)$ be an optimal allocation of buyer i with respect to a price vector $\mathbf{p}$. Then there exists a constant $c_i \geq 0$ such that for any other allocation $\mathbf{y}_i'$ for buyer i, we have*

$$\textstyle\sum_{j\in[m]} c_i p_j (y_{ij} - y_{ij}') \leq u_i(\mathbf{y}_i) - u_i(\mathbf{y}_i')$$

In particular, this implies that $c_i e_i \leq u_i(\mathbf{y}_i)$.

Proof. Let $L(\mathbf{x}_i, \lambda) = -u_i(\mathbf{x}_i) + \lambda(\mathbf{p} \cdot \mathbf{x}_i - e_i)$ be the Lagrangian associated with the problem (1) and

$$g(\lambda) = \inf_{\mathbf{x}_i \in [0,1]^m} \big(-u_i(\mathbf{x}_i) + \lambda(\mathbf{p} \cdot \mathbf{x}_i - e_i)\big)$$

be the dual function. Since $u_i(\cdot)$ is concave and the constraint is affine, it is not hard to see that the problem (1) satisfies strong duality (Boyd and Vandenberghe 2004). Let c_i be the optimal dual solution of $g(\lambda)$. Then

$$-u_i(\mathbf{y}_i) = g(c_i) \leq L(\mathbf{y}_i, c_i) \leq -u_i(\mathbf{y}_i)$$

where the first equality follows from strong duality, the second inequality follows by definition, and the third inequality follows by $c_i \geq 0$ and $\mathbf{p} \cdot \mathbf{x}_i \leq e_i$. Hence

$$L(\mathbf{y}_i, c_i) = g(c_i) \leq L(\mathbf{y}_i', c_i)$$

for any other allocation vector $\mathbf{y}_i' \in [0,1]^m$. □

In the above claim, c_i gives a lower bound on the bang-per-buck ratio of buyer i's optimal consumption, where the *bang-per-buck* ratio is given by $u_i(\mathbf{y}_i)/e_i$.

A central definition employed in the paper is the following weak gross substitute (c.f. (Mas-Colell, Whinston, and Green 1995)). We will assume that utility functions are differentiable on $[0,1]^m$.

Definition 3 (Weak gross substitute)**.** A utility function $u_i(\cdot)$ is said to have the *weak gross substitute (*`WGS`*)* property if the demand set $D_i(\mathbf{p}, e_i)$ of buyer i contains a unique allocation, that is, if the maximization of $u_i(\mathbf{x})$ subject to the endowment constraint, as a function of prices, yields demand functions $d_{ij}(\mathbf{p})$ from each buyer i to item j that is differentiable, and $\frac{\partial d_{ij}}{\partial p_k} \geq 0,\ \forall k \neq j$.

`WGS` property ensures that increasing the prices of some items will not cause a buyer to reduce his consumption of an item whose price has not been changed. It is also not hard to see that equilibrium prices are unique. The family of constant elasticity of substitution (CES) utility functions, i.e., $u(\mathbf{x}) = (\sum_{j\in[m]} \alpha_j x_j^\rho)^{1/\rho}$ where $0 \leq \rho < 1$, are examples that satisfy the above definition.

Incentive Ratio

In a market M, we consider a buyer $i \in [n]$ who attempts to improve his market allocation through misreporting his utility function and endowment. Let U_i denote the class of utility functions that i can feasibly report. Given a reported profile P, which consists of a vector of utility functions and a vector of endowments of all buyers, let $\mathbf{x}_i(P)$ denote the equilibrium allocation of buyer i. If $u_i(\cdot) \in U_i$ and e_i are buyer i's private true utility function and endowment, respectively, then the *incentive ratio* of buyer i in the market M is defined to be

$$\zeta_i^M = \max_{u_{-i};\, e_{-i}} \max_{u_i' \in U_i;\, e_i' \leq e_i} \frac{u_i\big(\mathbf{x}_i(u_i', e_i'; u_{-i}, e_{-i})\big)}{u_i\big(\mathbf{x}_i(u_i, e_i; u_{-i}, e_{-i})\big)}$$

where we assume that one cannot report a budget beyond his true endowment, i.e., $e_i' \leq e_i$. In the above definition, the denominator is the utility of buyer i when he bids u_i and e_i truthfully, and the numerator is the largest possible utility of buyer i when he unilaterally changes his bid given all other buyers' bids unchanged. The incentive ratio of buyer i is defined as the maximum ratio over all possible utilities and endowments of other buyers; it thus implies an upper bound on utility gain from manipulation. The incentive ratio of the market M with respect to a given class of utility functions is then defined as $\zeta^M = \max_{i\in[n]} \zeta_i^M$. Clearly, in a market with incentive ratio ζ, truthfully reporting is $1/\zeta$-approximate Nash equilibrium since no buyers can improve his own utility by a factor ζ via unilaterally manipulation.

Weak Gross Substitute

In this section, we analyze the incentive ratio of utility functions that satisfy the `WGS` condition. We first demonstrate the following lemma, which says that if prices are changed, a buyer will spend no less money on those items whose prices are decreased.

Lemma 3. *Given a utility function $u(\cdot)$ that satisfies the* `WGS` *condition, let $\mathbf{p}$ and $\mathbf{p}'$ be two price vectors and $\mathbf{x} \in D(\mathbf{p}, e)$ and $\mathbf{x}' \in D(\mathbf{p}', e)$ be the corresponding optimal demands. Let $S = \{j \in [m] \mid p_j > p_j'\}$ denote the set of items whose prices are decreased from $\mathbf{p}$ to $\mathbf{p}'$. Then $\sum_{j\in S} x_j p_j \leq \sum_{j \in S} x_j' p_j'$ and $\sum_{j\notin S} x_j' p_j' \leq \sum_{j\notin S} x_j p_j$.*

Proof. Define a price vector $\mathbf{p}^* = (p_j^*)_j$ as $p_j^* = \min\{p_j, p_j'\}$ and consider an optimal demand $\mathbf{x}^* \in D(\mathbf{p}^*, e)$. Note here that $p_j^* = p_j' < p_j$, for all $j \in S$; otherwise $p_j^* = p_j$. That is, the prices of all item $j \in S$ decrease from p to p^* while others' prices remain the same. For all $j \in [m] \setminus S$, applying the `WGS` property on $u(\cdot)$ with prices $\mathbf{p}$ and $\mathbf{p}^*$, we have $x_j^* \leq x_j$ and $p_j^* x_j^* = p_j x_j^* \leq p_j x_j$.

Due to the fact that $\sum_{j\in[m]} x_j p_j = e = \sum_{j\in[m]} x_j^* p_j^*$, we have $\sum_{j\in S} x_j^* p_j' = \sum_{j\in S} x_j^* p_j^* \geq \sum_{j\in S} x_j p_j$. Using the WGS condition again with prices $\mathbf{p}'$ and $\mathbf{p}^*$, we get $x_j^* \leq x_j'$ for all $j \in S$. Combining these two inequalities, we have $\sum_{j\in S} x_j p_j \leq \sum_{j\in S} x_j' p_j'$. This completes the proof. □

Our main result of the section is the following.

Theorem 4. *For any market M with utility functions that satisfy the WGS condition, the incentive ratio of the market is at most 2, i.e., $\zeta^{WGS} \leq 2$.*

Proof. Without loss of generality, we will consider buyer 1 and show that $\zeta_1^M \leq 2$. Let e_1' denote buyer 1's reported budget. For any fixed bids of other buyers, let $(\mathbf{p}, \mathbf{x})$ be any equilibrium when buyer 1 bids truthfully and $(\mathbf{p}', \mathbf{x}')$ be any equilibrium when he bids strategically. It suffices to prove that $u_1(\mathbf{x}_1') \leq 2u_1(\mathbf{x}_1)$. Let S denote the set of items whose price are decreased from $\mathbf{p}$ to $\mathbf{p}'$, i.e., $S = \{j \in [m] \mid p_j > p_j'\}$, and let $T = [m] \setminus S$ denote the set of items whose prices are not decreased. By Prop. 2, there exists a constant c such that for buyer 1,

$$u_1(\mathbf{x}_1') - u_1(\mathbf{x}_1) \leq \sum_{j\in[m]} c \cdot p_j \cdot (x_{1j}' - x_{1j}) \quad (2)$$
$$= \sum_{j\in T} c \cdot p_j \cdot (x_{1j}' - x_{1j}) + c \cdot \sum_{j\in S} p_j (x_{1j}' - x_{1j})$$

Note that $\mathbf{x}_i \in D_i(\mathbf{p}, e_i)$ and $\mathbf{x}_i' \in D_i(\mathbf{p}', e_i)$, for any buyer $i \neq 1$. By Lem. 3, they spend more money on the items in S as their prices are decreased. In aggregate,

$$\sum_{j\in S} p_j - \sum_{j\in S} x_{1j} p_j = \sum_{i\neq 1} \sum_{j\in S} x_{ij} p_j$$
$$\leq \sum_{i\neq 1} \sum_{j\in S} x_{ij}' p_j' = \sum_{j\in S} p_j' - \sum_{j\in S} x_{1j}' p_j'$$

which implies $\sum_{j\in S} x_{1j} p_j \geq \sum_{j\in S} x_{1j}' p_j' + \sum_{j\in S} (p_j - p_j')$. Therefore,

$$\sum_{j\in S} x_{1j}' p_j - \sum_{j\in S} x_{1j} p_j$$
$$\leq \sum_{j\in S} x_{1j}' p_j - \sum_{j\in S} x_{1j}' p_j' - \left(\sum_{j\in S} p_j - \sum_{j\in S} p_j'\right)$$
$$\leq \sum_{j\in S} (x_{1j}' - 1)(p_j - p_j') \leq 0$$

In addition, for any $j \in T$, since $p_j < p_j'$,

$$\sum_{j\in T} c \cdot p_j \cdot (x_{1j}' - x_{1j}) \leq \sum_{j\in T} c \cdot p_j x_{1j}'$$
$$\leq \sum_{j\in T} c \cdot p_j' x_{1j}' \leq c \cdot e_1' \leq c \cdot e_1 \leq u_1(\mathbf{x}_1) \quad \text{(by Prop. 2)}$$

Substituting the above inequalities to Eq. (2) yields $u_1(\mathbf{x}_1') - u_1(\mathbf{x}_1) \leq u_1(\mathbf{x}_1)$. This completes the proof. □

Note that the theorem applies to all utility functions that satisfy the WGS condition. In addition, the ratio 2 given by the theorem is tight (see Example 1).

An Example of Unbounded Incentive Ratio

We describe a situation where incentive ratio could be arbitrarily large for some additive piecewise linear utilities

Example 5. *We consider a market with two items and two buyers where both buyers have linear utilities. Their utilities and endowments are as follows.*

Table 1: An example for unbound incentive ratios

	u_{i1}	u_{i2}	e_i
buyer 1	$\frac{x_{11}}{2}$	$\frac{x_{12}}{2}$	ϵ
buyer 2	x_{21}	$h(x_{22})$	$1-\epsilon$

Here $h(x)$ is a piecewise linear and concave function defined as below: $h(x) =$

$$\begin{cases} kx & \text{if } x \leq t \\ (\frac{k-1}{\delta}t + k)x - \frac{(k-1)x^2}{2\delta} - \frac{(k-1)t^2}{2\delta} & \text{if } t < x \leq t+\delta \\ x + (k-1)t + \frac{(k-1)\delta}{2} & \text{if } x > t+\delta \end{cases}$$

where $k = \frac{1-\frac{\epsilon}{2}}{\frac{\epsilon}{2}}$ and $t = \frac{1-\epsilon}{1-\frac{\epsilon}{2}}$, and δ is a sufficiently small number. The following figure shows an example of $h(x)$ when $\epsilon = 0.2$ (thus, $k = 9$ and $t = \frac{8}{9}$) and $\delta = \frac{t}{100} = \frac{8}{900}$.

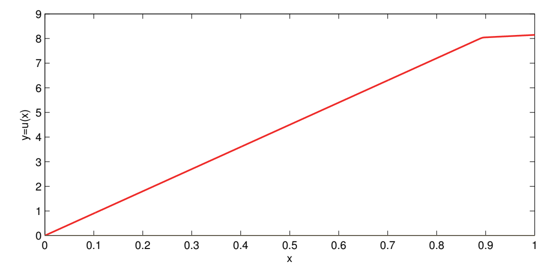

Figure 1: An illustration of $h(x)$: the marginal utility diminishes quickly after the threshold t.

Note that the utility function of the second buyer does not satisfy the WGS condition. Indeed, when the first item's price decreases from $\frac{1}{2}$ to $\frac{\epsilon}{2}$ and the second item's price increases from $\frac{1}{2}$ to $1-\frac{\epsilon}{2}$, the second buyer's demand for the first item decreases, contradicting the WGS condition.

If the first buyer bids truthfully, the equilibrium price vector is $(\frac{1}{2}, \frac{1}{2})$ and his utility is ϵ. But if he misreports his utility function to be $\left(\frac{\epsilon}{2}x_{11}, (1-\frac{\epsilon}{2})x_{12}\right)$, the equilibrium price vector becomes $(\frac{\epsilon}{2}, 1-\frac{\epsilon}{2})$ and he gets the first item with a utility of at least $\frac{1}{2}$. Thus, his incentive ratio is at least $\frac{1}{2\epsilon}$, which is unbounded when ϵ approaches 0.

Parameterized Study on Strategic Behaviors

In this section, we conduct a parameterized study on incentive ratio to identify some sufficient conditions for a market where buyers have little incentive to deviate from truthfully reporting. In particular, we will consider how initial endowment and maximum share might affect a player's incentive to manipulate in the game. To prove the results, we need the homogeneity assumption.

Definition 4 (Homogeneity)**.** A utility function $u(\mathbf{x})$ is homogeneous of degree k if for all $c \in \mathbb{R}$, $u(c \cdot \mathbf{x}) = c^k \cdot u(\mathbf{x})$.

Homogeneity and WGS together give us a more precise characterization of a buyer's consumption after prices change, as the following lemma demonstrates.

Lemma 6. *Assume that $u(\cdot)$ is* WGS *and homogeneous. Let $\mathbf{p}$ and $\mathbf{p}'$ be two price vectors and $\mathbf{x} \in D(\mathbf{p}, e)$ and $\mathbf{x}' \in D(\mathbf{p}', e)$ be the corresponding optimal demands. For any $q > 0$, let $S(q) = \{j \in [m] \mid q \cdot p_j > p'_j\}$. Then $\sum_{j\in S(q)} x_j p_j \le \sum_{j \in S(q)} x'_j p'_j$. Equivalently, if we denote $T(q) = [m] \setminus S(q)$, then $\sum_{j\in T(q)} x'_j p'_j \le \sum_{j\in T(q)} x_j p_j$.*

Proof. Let $\mathbf{p}'' = \mathbf{p}'/q$ and $\mathbf{x}'' = q \cdot \mathbf{x}'$. Since $u(\cdot)$ is homogeneous, $u(\mathbf{x}'') = q^k \cdot u(\mathbf{x}')$. Since $\mathbf{x}' \in D(\mathbf{p}', e)$ and $\mathbf{x}''$ is a feasible allocation given price $\mathbf{p}''$, we know that $\mathbf{x}'' \in D(\mathbf{p}'', e)$. By the definition of $\mathbf{p}''$, for all item $j \in S(q)$, $p''_j = p'_j/q < p_j$; and for all item $j \notin S(q)$, $p''_j \ge p_j$. Applying Prop. 3 with prices $\mathbf{p}$ and $\mathbf{p}''$ and the corresponding optimal allocations $\mathbf{x}$ and $\mathbf{x}''$ gives $\sum_{j\in S} x''_j p''_j \ge \sum_{j\in S} x_j p_j$ where $S = \{j \in [m] \mid p_j > p''_j\} = S(q)$. The lemma follows the fact that $x''_j = q x'_j$ and $p''_j = p'_j/q$. □

Endowment

We show a negative relation between a buyer's incentive ratio and his money endowment with the following theorem.

Theorem 7. *For any market M with homogeneous utilities that satisfy the* WGS *condition, consider any buyer i with an initial endowment of e_i, we have $\zeta_i^M \le 2 - e_i$.*

Proof. W.l.o.g., it suffices to show $\zeta_1 \le 2 - e_1$. Let $(\mathbf{p}, \mathbf{x})$ and $(\mathbf{p}', \mathbf{x}')$ be market equilibria when buyer 1 bids truthfully and strategically, respectively. We will show that $u_1(\mathbf{x}'_1) \le (2 - e_1) \cdot u_1(\mathbf{x}_1)$. Let $R(q_k) = \{j \in [m] \mid p'_j = q_k \cdot p_j\}$. Divide all items into a collection of subsets such that: $[m] = \bigcup_{k=1}^t R(q_k)$ where $q_1 > q_2 > \cdots > q_t$. Let $\gamma_k = \sum_{j\in R(q_k)} p_j$ and $\gamma'_k = \sum_{j\in R(q_k)} p'_j$ be the sum of the prices of the items in $R(q_k)$ with respect to p and p', respectively. Further, define $\beta_k = \sum_{j\in R(q_k)} p_j x_{1j}$ and $\beta'_k = \sum_{j\in R(q_k)} p'_j x'_{1j}$ to be the amount of money that buyer 1 spends on the set of items $R(q_k)$ in the consumptions $\mathbf{x}_1$ and $\mathbf{x}'_1$, respectively. It follows that $\sum_{k=1}^t \beta_k = e_1$ and $\sum_{k=1}^t \beta'_k = e'_1$, where e'_1 is the amount of endowment that buyer 1 spends in $\mathbf{x}'$.

Next, by Prop. 2, there exists a constant c such that

$$
\begin{aligned}
u_1(\mathbf{x}'_1) - u_1(\mathbf{x}_1) &\le \textstyle\sum_{j\in[m]} c \cdot p_j (x'_{1j} - x_{1j}) \\
&= \textstyle\sum_{k=1}^t \sum_{j\in R(q_k)} c \cdot \left(\frac{p'_j x'_{1j}}{q_k} - p_j x_{1j}\right) \\
&= \textstyle\sum_{k=1}^t c \cdot \left(\frac{\beta'_k}{q_k} - \beta_k\right) \triangleq \Delta \qquad (3)
\end{aligned}
$$

Thus, to have an upper bound on the utility gain $u_1(\mathbf{x}'_1) - u_1(\mathbf{x}_1)$, it suffices to bound Δ. Particularly we will try to identify the constraints for and optimize over the sequence $\{\beta'_k\}$, while assuming other parameters are already fixed.

For any $k = 1, \ldots, t-1$, note that $T(q_k) = \{j \in [m] \mid q_k \cdot p_j \le p'_j\} = \bigcup_{\ell=1}^k R(q_\ell)$. By Lem. 6, for all buyers $i \neq 1$, they spend less money on the items in $T(q_k)$ after prices are changed from p to p'. We therefore have

$$
\begin{aligned}
\textstyle\sum_{\ell=1}^k (\gamma'_\ell - \beta'_\ell) &= \textstyle\sum_{j\in T(q_k)} p'_j - \sum_{j\in T(q_k)} p'_j x'_{1j} \\
&\le \textstyle\sum_{j\in T(q_k)} p_j - \sum_{j\in T(q_k)} p_j x_{1j} \\
&= \textstyle\sum_{\ell=1}^k (\gamma_\ell - \beta_\ell)
\end{aligned}
$$

This implies that

$$
\textstyle\sum_{\ell=1}^k \beta'_\ell \ge \sum_{\ell=1}^k (\beta_\ell + \gamma'_\ell - \gamma_\ell). \qquad (4)
$$

for any $k = 1, \ldots, t-1$. Note as well that the inequality holds in (4) when $k = t$.

Now we are ready to estimate Δ. Since $\{q_k\}$ is a decreasing sequence, Δ can be bounded by the case when the vector $(\beta'_1, \beta'_2, \ldots, \beta'_t)$ is lexicographically minimized, subject to the set of constraints in (4). In other words, the utility gain is maximized when less money is spent on those items whose price are increased more. However, under the set of constraints in (4), $(\beta'_1, \beta'_2, \ldots, \beta'_t)$ is lexicographically minimized when $\beta'_k = \beta_k + \gamma'_k - \gamma_k,\ \forall\ k = 1 \ldots t$. And this assignment also satisfies $\sum_{k=1}^t \beta'_k = \sum_{k=1}^t (\beta_k + \gamma'_k - \gamma_k)$. Summing overall, Δ/c in Eq. (3) becomes:

$$
\begin{aligned}
&\textstyle\sum_{k=1}^t \left(\frac{\beta'_k}{q_k} - \beta_k\right) \le \sum_{k=1}^t \left(\frac{\beta_k + \gamma'_k - \gamma_k}{q_k} - \beta_k\right) \\
=\ & \textstyle\sum_{k=1}^t \frac{\gamma_k(\beta_k + \gamma'_k - \gamma_k) - \beta_k q_k \gamma_k}{q_k \gamma_k} \\
=\ & \textstyle\sum_{k=1}^t (\gamma_k - \beta_k)(\gamma'_k - \gamma_k)/\gamma'_k \quad (\text{by } q_k\gamma_k = \gamma'_k) \\
\le\ & \textstyle\sum_{k=1}^t (\gamma_k - \beta_k)(\beta_k + \gamma'_k - \gamma_k)/\gamma'_k \quad (\text{by } \gamma_k \ge \beta_k) \\
\le\ & \textstyle\sum_{k=1}^t (\gamma_k - \beta_k) \cdot \sum_{k=1}^t (\beta_k + \gamma'_k - \gamma_k) / \sum_{k=1}^t \gamma'_k \\
=\ & \left(\textstyle\sum_{k=1}^t \gamma_k - \sum_{k=1}^t \beta_k\right) \cdot \sum_{k=1}^t \beta'_k \le (1 - e_1) \cdot e'_1
\end{aligned}
$$

The second last inequality follows from repeatedly applying the following Fact 8. Therefore, Eq. (3) becomes $u_1(\mathbf{x}'_1) - u_1(\mathbf{x}_1) \le c e'_1 \cdot (1 - e_1) \le c e_1 \cdot (1 - e_1) \le u_1(\mathbf{x}_1) \cdot (1 - e_1)$, where the last inequality is by Prop. 2. □

The following fact is used in the above proof. It can be verified to be equivalent to $(a_1 b_2 - a_2 b_1)^2 \ge 0$.

Fact 8. *Assume that $a_1 + b_1$ and $a_2 + b_2$ are positive, then*

$$
\frac{a_1 b_1}{a_1 + b_1} + \frac{a_2 b_2}{a_2 + b_2} \le \frac{(a_1 + a_2)(b_1 + b_2)}{a_1 + b_1 + a_2 + b_2}
$$

Maximum Share

We present a positive relation between a buyer's incentive ratio and his maximum share, defined as the maximum allocation among all items that the buyer consumes: $\alpha_i = \max_{(x,p)} \max_{j\in[m]} x_{ij}$ where the maximum is taken over all market equilibria. The following claim suggests that if a buyer dominates an item, then there may be a larger room for him to improve his utility by manipulation.

Theorem 9. *For any market M with homogeneous utilities that satisfy the* WGS *condition, the incentive ratio of any buyer i satisfies $\zeta_i^M \le 1 + \alpha_i$, where α_i is the maximum share of the buyer.*

Proof. It suffices to show $u_1(\mathbf{x}_1') \le (1+\alpha_1)\cdot u_1(\mathbf{x}_1)$. Recall Eq. (3) from Thm. 7:

$$u_1(\mathbf{x}_1') - u_1(\mathbf{x}_1) \le \sum_{k=1}^{t} c \cdot \left(\frac{\beta_k'}{q_k} - \beta_k\right) \qquad (5)$$

Let s denote the maximum index where $q_k \ge 1$ for any $k \le s$; thus, the prices of all items in $T(q_s)$ are increased from p to p'. Note that

$$\begin{aligned}\sum_{k=1}^{s} c \cdot \left(\frac{\beta_k'}{q_k} - \beta_k\right) &\le \sum_{k=1}^{s} c \cdot (\beta_k' - \beta_k) \\ &\le \sum_{k=s+1}^{t} c \cdot (\beta_k - \beta_k')\end{aligned}$$

The last inequality comes from Lem. 6. Substituting it to Eq. (5), we have

$$\begin{aligned}&u_1(\mathbf{x}_1') - u_1(\mathbf{x}_1) \le \sum_{k=s+1}^{t} c \cdot \left(\frac{\beta_k'}{q_k} - \beta_k + \beta_k - \beta_k'\right) \\ &= \sum_{k=s+1}^{t} c \cdot \beta_k' \cdot \frac{1-q_k}{q_k} \triangleq \Delta\end{aligned}$$

We identify constraints for the set $\{\beta_k'\}_{k=s+1}^{t}$. By Lem. 6, we know that all buyers $i \ne 1$ spend more money on the items in $S(q_k) = \bigcup_{\ell=k+1}^{t} R(q_\ell)$, for any $k = s, \dots, t-1$. Therefore (following a similar argument for Eq. (4)),

$$\begin{aligned}&\sum_{\ell=k+1}^{t} (\gamma_\ell' - \beta_\ell') \ge \sum_{\ell=k+1}^{t} (\gamma_\ell - \beta_\ell) \\ \Rightarrow \quad &\sum_{\ell=k+1}^{t} \beta_\ell' \le \sum_{\ell=k+1}^{t} (\beta_\ell + \gamma_\ell' - \gamma_\ell). \qquad (6)\end{aligned}$$

for any $k = s, \dots, t-1$ in particular.

Since $\{\frac{1-q_k}{q_k}\}$ is an increasing sequence, Δ can be bounded by the case when the vector $(\beta_t', \dots, \beta_{s+1}')$ is lexicographically maximized, subject to the set of constraints (6). On the other hand, $(\beta_t', \dots, \beta_{s+1}')$ is lexicographically maximized with these constraints when $\beta_k' = \beta_k + \gamma_k' - \gamma_k, \forall k = s+1, \dots, t$. Therefore,

$$u_1(\mathbf{x}_1') - u_1(\mathbf{x}_1) \le c \cdot \sum_{k=s+1}^{t} (\beta_k + \gamma_k' - \gamma_k) \cdot \frac{1-q_k}{q_k} \quad (7)$$

Note that when $k > s$, we have

$$\begin{aligned}&\frac{\beta_k + \gamma_k' - \gamma_k}{\beta_k} \cdot \frac{1-q_k}{q_k} = \frac{\beta_k + \gamma_k' - \gamma_k}{\beta_k} \cdot \frac{\gamma_k - \gamma_k'}{\gamma_k'} \\ \le \quad &\frac{\alpha_1\gamma_k + \gamma_k' - \gamma_k}{\alpha_1\gamma_k} \cdot \frac{\gamma_k - \gamma_k'}{\gamma_k'} \le \alpha_1\end{aligned}$$

The first inequality follows from the facts that $\gamma_k' < \gamma_k$ (as $k > s$) and α_1 is the largest share of buyer 1 (hence $\beta_k \le \alpha_1\gamma_k$), and the last one follows by rearranging the term to the normal form $(1-\alpha_1)\gamma_k^2 + \gamma_k\gamma_k'(\alpha_1^2+\alpha_1-2) + {\gamma_k'}^2 \ge 0$ and the fact that $1-\alpha_1 \ge 0$ and $(\alpha_1^2+\alpha_1-2)^2 - 4(1-\alpha_1) \le 0$. Therefore Eq. (7) becomes

$$u_1(\mathbf{x}_1') - u_1(\mathbf{x}_1) \le c \cdot \sum_{k=s+1}^{t} \alpha_1 \cdot \beta_k \le \alpha_1 c e_1 \le \alpha_1 u_1(\mathbf{x}_1)$$

where recall that $\sum_{k=1}^{t} \beta_k = e_1$ and the last inequality follows from Prop. 2. □

Collusive Strategic Behavior

In this section, we examine strategic plays from the viewpoint of collusive behavior, that is, whether buyers have more incentives to form a collusion instead of manipulating individually. We first generalize the definition of incentive ratio to collusion. Let S be a coalition of buyers. The incentive ratio of S is defined as the largest utility gain of an individual in S given that no one in S is worse off.

The following example shows that the utility gain of a coalition can be unbounded, even in a large market.

Example 10. *Consider a market with n buyers and m items where the utility of each buyer is additive. Buyers $1, 2, 3$ are interested only in the first four items while all other buyers are interested in the rest items. The profiles of the first three buyers are shown as follows.*

Table 2: An example for collusive strategic behavior

	u_{i1}	u_{i2}	u_{i3}	u_{i4}	e_i
buyer 1	$\frac{x_{11}}{n}$	0	$\frac{x_{13}}{4}$	0	$\frac{2}{n^3}$
buyer 2	0	$(1-\frac{3}{2n})x_{22}$	$\frac{x_{23}}{2n}$	$\frac{x_{24}}{n}$	$\frac{1}{n} - \frac{1}{n^2}$
buyer 3	$\frac{2\cdot x_{31}}{n+2}$	0	0	$\frac{n\cdot x_{34}}{n+2}$	$\frac{1}{n^2}$

The equilibrium price of the first four items is $(\frac{2}{n^3}, \frac{1}{n} - \frac{3}{2n^2}, \frac{1}{2n^2}, \frac{1}{n^2})$. Buyer 1 gets the 1st item with utility $u_1(\mathbf{x}_1) = \frac{1}{n}$, and buyer 2 gets the 2nd and 3rd items with utility $u_2(\mathbf{x}_2) = 1 - \frac{1}{n}$. Suppose that buyer 1 and 2 form a collusion by bidding $u_1'(\mathbf{x}_1) = x_3$ and $u_2'(\mathbf{x}_2) = (\frac{1}{2} - \frac{1}{n}) \cdot x_2 + \frac{x_4}{2}$. Then the equilibrium price becomes $(\frac{1}{n^2}, \frac{1}{2n} - \frac{1}{n^2}, \frac{2}{n^3}, \frac{1}{2n})$. Now buyer 1 gets the 3rd item with utility $u_1(\mathbf{x}_1') = \frac{1}{4}$, and buyer 2 gets the 2nd and 4th item with utility $u_2(\mathbf{x}_2') = 1 - \frac{1}{2n}$. It can be seen that the incentive ratio of the coalition (precisely, buyer 1) is unbounded (even when n approaches infinity).

In the above example, one buyer in the collusion has a maximum share of 1, which means that the collusion dominate some items. If the dominant power of a collusion is restricted, then the incentive ratio of the collusion is also bounded. Let us put forward to the formal definition of the maximum share α_S of a collusion S. That is, $\alpha_S = \max_{(x,p)} \max_{j\in[m]} \sum_{i\in S} x_{ij}$ where the maximum is taken over all market equilibria.

The next theorem (whose proof is deferred to supplementary material), which is slightly weaker than Thm. 9 for the case when $|S| = 1$, gives an upper bound of the incentive ratio for a coalition.

Theorem 11. *For homogeneous utility functions that satisfy the* `WGS` *condition, the incentive ratio of a collusion S is at most $\frac{1}{1-\alpha_S}$, where α_S is the maximum share of the collusion.*

Remark. The above theorem shows that if a coalition has a very small share of items in a market, then they have little influence on market prices. Jackson and Manelli (1997) have explored a similar idea: the condition that reported economy approaches the true economy in a large market relies on the key assumption that agents believe that they can have little influence on market prices.

Conclusion

This paper provided an approach towards quantifying incentives in the market equilibrium mechanism in respect to strategic plays of participating buyers. We illustrated a tight bound for WGS within the Fisher market model and explored qualitative properties that may affect buyers' incentive for deviating from truthfully reporting. It is interesting to explore incentive ratio beyond Fisher market model. In particular, it seems promising to adopt the incentive ratio and our techniques in a Walrasian exchange market introduced in (Gul and Stacchetti 1999).

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