On Strategy-proof Allocation without Payments or Priors

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Abstract. In this paper we study the problem of allocating divisible items to selfish agents without payments. We assume no prior knowledge about the agents. The utility of an agent is additive. The social welfare of a mechanism is defined as the overall utility of all agents. This model is first defined by Guo and Conitzer[7]. Here we are interested in strategy-proof mechanisms that have a good *competitive ratio*, that is, those that are able to achieve social welfare close to the maximal social welfare in all cases. First, for the setting of n agents and m items, we prove that there is no $(1/m + \epsilon)$-competitive strategy-proof mechanism, for any $\epsilon > 0$. And, no mechanism can achieve a *competitive ratio* better than $4/\sqrt{n}$, when $m \geq \sqrt{n}$. Next we study the setting of two agents and m items, which is also the focus of [7]. We prove that the *competitive ratio* of any swap-dictatorial mechanism is no greater than $1/2 + 1/\sqrt{\lceil \log m \rceil}$. Then we give a characterization result: for the case of 2 items, if the mechanism is strategy-proof, symmetric and second order continuously differentiable, then it is always swap-dictatorial. In the end we consider a setting where an agent's valuation of each item is bounded by C/m, where C is an arbitrary constant. We show a mechanism that is $(1/2 + \epsilon(C))$-competitive, where $\epsilon(C) > 0$.

1 Introduction

The agenda of approximate mechanism design without money was first explicitly framed by Procaccia and Tennenholtz in their seminal paper[14], and can be traced back to the work on incentive compatible learning by Deckel et al. [4]. This line of research tries to study how to design truthful mechanisms when payment is not allowed. As noted by Schummer and Vohra[15], "there are many important environments where money cannot be used as a medium of compensation. This constraint can arise from ethical and/or institutional considerations." To this end, Procaccia and Tennenholtz suggests "approximation can be used to obtain strategyproofness without resorting to payments." The *approximation ratio* of a mechanism will be measured on how close it approximates an optimal solution. Following this idea, several models have been studied extensively, for instance, *facility game*[14,9,1], *classification*[10,11,12].

In this paper, we consider the following resource allocation problem: there are n agents, m heterogeneous, divisible items. Each agent has a private valuation over the items about which we have no prior knowledge. Their utility functions are linear. A *competitive* allocation mechanism tries to maximize the society's social welfare, that is, the sum of each agent's utility. The competitive ratio is measured in terms of the worst-case behavior of a mechanism compared to an optimal allocation. We are interested in strategy-proof and at the same time *competitive* mechanisms. While our model is very simple, it also sounds natural. When a central agency tries to allocate various public resources to people efficiently, it faces a similar problem as described in our model. Besides, some resources are divisible in nature, for instance, water, bandwidth, etc. Also, when the items are indivisible and the agents are risk-neutral, the expectation of a randomized mechanism in this case corresponds to a deterministic mechanism in our model.

The problem of resource allocation has been studied in algorithmic game theory on various aspects. Ramesh Johari[8] discusses the problem of allocating an infinitely divisible resource of a fixed capacity to various users, who have their own utility functions and pay money to obtain resources. He gives a proportional mechanism that is quite efficient. There's also work on the allocation of indivisible resources without payments. Eric Budish [2] studies a similar combinatorial assignment problem and surveys existing allocation mechanisms. Szilvia Pápai [13] shows that strategy-proof combined with conditions like onto, non-bossiness, etc, can only lead to dictatorship.

Our Results In the general setting, if we consider an even allocation, that is, allocating each item equally between agents, or a biased plan to allocate all the resources to a single designated agent, the competitive ratios are both $1/m$. In both mechanisms the ratio becomes very small as m grows. Thus the first question arises as:

Question 1. Is there a c-competitive strategy-proof mechanism for any number of agents and items, where $c > 0$?

As it turns out, the answer is negative. We give the following result: there does not exist a $(1/m + \epsilon)$-competitive strategy-proof mechanism, for any $\epsilon > 0$. By a similar technique we also show that the competitive ratio of a strategy-proof mechanism is less than $4/\sqrt{n}$, when $m \geq \sqrt{n}$. This result stands in contrast with the VCG mechanism[16,3,6], which gives an optimal allocation if payments can be used in our model.

Having dealt with the multi-agents setting, next we come to the setting of two agents and any number of items, which is also the focus of Guo and Conitzer[7]. There they used swap-dictatorial(SD) as a basic tool to design strategy-proof mechanisms. The idea of SD is, each agent has some chance to be the dictator, choosing his preferred allocation from a predefined set. The final allocation will be the weighted sum from each agent's choice. We find two interesting results about swap-dictatorial mechanisms. The first is a somewhat surprising link between SD and strategy-proof: In the setting of 2 agents and 2 items, when a mechanism is symmetric and second order continuously differentiable, then strategy-proof coincides with swap-dictatorial. Since items are divisible, the model we are dealing with is inherently a continuous one. The tools from calculus provide us a way to interpret and characterize the strategy-proof condition, making the problem much simpler to handle. The second result is that the competitive ratio of an SD mechanism is always less than $1/2 + 1/\sqrt{\lceil \log m \rceil}$. In particular this implies when there are too many items, SD is not much better than an even allocation. We remark that when the number of items is small, it is still possible to obtain competitive SD mechanisms. The linear increasing-price mechanism is just swap-dictatorial[7]. In that paper it is also proved that LIP is 0.828-competitive when there are 2 items, nearly reaching their established upper bound of 0.841.

Given the negative result on swap-dictatorial mechanisms, it is natural to ask the second question:

Question 2. Is there a c-competitive strategy-proof mechanism for 2 agents, any number of items, where $c > 1/2$?

The question is still *open*. And the only result is that c is smaller than 0.841, as we just mentioned above. Note that our characterization between strategy-proof and SD, if generalized to the any number of items and any mechanism, will give a negative answer to the above question.

Since it appears hard to design a strategy-proof mechanism that beats the 0.5 ratio, and it seems unreasonable to assume that agents' valuations are completely unrestricted, we come to a bounded-valuation setting when an agent's valuation cannot be strongly biased. Here we manage to demonstrate a swap-dictatorial mechanism that is competitive as well, giving a positive answer to Question 2 in a restricted domain.

2 Preliminaries and the Model

We briefly describe our model here, the reader may refer to [7] for more details and discussions.

There are m items, each with capacity 1. These items are allocated to n agents, who keep their valuations on the items in private. The valuation is a *vector* $\boldsymbol{v} = (v_1, \ldots, v_m) \in [0,1]^m$, where $\sum_{i=1}^m v_i = 1$. The normalization says when an agent gets all the resources, he or she gains one unit of utility. Let $\mathbf{V}$ be the set of valuation vectors. A *valuation matrix* is an $n \times m$ matrix V where each row is a *valuation vector*. We use $\boldsymbol{v}_i$ to denote the i-th row of V, v_{ij} to denote the j-th component of $\boldsymbol{v}_i$. Let $\mathbf{U}$ be the space of valuation matrices. An *allocation vector* $\boldsymbol{o} = (o_1, \ldots, o_m) \in [0,1]^m$. An *allocation matrix* is an $n \times m$ matrix $O = (o_{ij})_{n \times m}$ where $o_{ij} \in [0,1]$ indicates the fraction of item j allocated to agent i, for all $1 \leq i \leq n$, $1 \leq j \leq m$. And $\sum_{i=1}^n o_{ij} = 1$, for $1 \leq j \leq m$, i.e., all of the items are allocated. Let $\mathbf{O}$ be the space of allocation matrices.

A *deterministic payment-free* mechanism is a function $M : \mathbf{U} \to \mathbf{O}$. Let $g_i(\boldsymbol{x}, O)$ be the i-th agent's *utility* under allocation O when his or her valution is $\boldsymbol{x}$. g_i is additive, that is, $g_i(\boldsymbol{x}, O) = \sum_{j=1}^m x_j \cdot o_{ij}$. Let $V(i, \boldsymbol{x})$ be the matrix obtained from substituting the i-th row vector of V by $\boldsymbol{x}$.

M is called *strategy-proof*, if for any *valuation matrix* V, *valuation vector* $\boldsymbol{x}$, $1 \le i \le n$, $g_i(\boldsymbol{v_i}, M(V(i, \boldsymbol{v}_i))) \ge g_i(\boldsymbol{v}_i, M(V(i, \boldsymbol{x})))$. In other words, no agent benefits by misreporting his *valuation vector*.

When there are only two agents, for ease of notation, we define the mechanism function as $M : \mathbf{V}^2 \to [0,1]^m$, since it's apparent that the other agent gets $\mathbf{1} - M(\boldsymbol{v}_1, \boldsymbol{v}_2)$, where $\mathbf{1}$ denotes a vector whose components are all 1. [1]

The *social welfare* is defined as $\sum_{i=1}^{n} g_i(\boldsymbol{v_i}, O)$, that is, the sum of all agents' utilities. The *optimal social welfare* $\gamma(V)$ is the social welfare under an *optimal allocation*, which ideally allocates each item to an agent that values it highest. We measure the competitiveness of a *strategy-proof* mechanism by comparing its achieved *social welfare* to the *optimal social welfare*. More formally, define the *competitive ratio* of a *strategy-proof* mechanism as

$$\min_{V \in \mathbf{U}} \frac{\sum_{i=1}^{n} g_i(\boldsymbol{v_i}, M(V))}{\gamma(V)}$$

We say that a *strategy-proof* mechanism is α-competitive, if its *competitive ratio* is at least α.

We point out here that randomness does not help provide a more *competitive* mechanism in this model. For if there is an α-competitive strategy-proof randomized mechanism M', then taking the expected outcome of M' gives us a deterministic strategy-proof mechanism that is also α-competitive. However, randomness is still useful for describing a mechanism, as we'll see below.

Definition 1 (Guo, Conitzer[7]). *A mechanism is symmetric if it satisfies:*

1. Symmetric over the agents*: if by swapping the valuations of two agents, their allocations are also swapped correspondingly.*
2. Symmetric over the items*: if by swapping the valuations of two items by each agent, the allocations for these two items are also swapped.*

Let P_{ij} be a permutation matrix that permutes row(or column) i, j. The following proposition is a direct translation of the symmetry condition.

Proposition 1. *A symmetric mechanism M satisfies:*

1. $M(P_{ij}V) = P_{ij} \cdot M(V)$.
2. $M(VP_{ij}) = M(V) \cdot P_{ij}$.

An important property of symmetric mechanisms is the following:

Proposition 2 (Guo, Conitzer[7]). *For any strategy-proof mechanism with competitive ratio α, there is a symmetric mechanism with competitive ratio at least α.*

Next we introduce the family of *swap-dictatorial*[7][2] mechanisms for two agents.

[1] We will always use $\boldsymbol{c}$ to denote a constant vector whose components are all c.

[2] We abbreviate swap-dictatorial by SD sometimes.

Definition 2. *Let D_1, D_2 be two sets of allocation vectors, $\boldsymbol{v}_1, \boldsymbol{v}_2$ be two valuation vectors. For $i = 1, 2$, let $f_i : \mathbf{V} \to D_i$ be a function such that $f_i(\boldsymbol{v}) \in \operatorname{argmax}_{\boldsymbol{o} \in D_i} \boldsymbol{v} \cdot \boldsymbol{o}$, for any $\boldsymbol{v} \in \mathbf{V}$. A swap-dictatorial mechanism M determined by D_1, D_2 is defined as $M(\boldsymbol{v}_1, \boldsymbol{v}_2) = (f_1(\boldsymbol{v}_1) + \mathbf{1} - f_2(\boldsymbol{v}_2))/2$.*

There is another intuitive description of swap-dictatorial mechanism: with probability 0.5, agent i becomes the dictator and chooses an allocation vector from D_i to selfishly maximize his or her own welfare, leaving the rest to the second agent. The expected outcome will be the resulted allocation. Note that while this description uses randomness, SD is actually a deterministic mechanism.

A swap-dictatorial mechanism is strategy-proof, which can be verified from the definitions. Intuitively, an agent's utility comes from two parts: one from being the dictator, here there is no incentive to lie; the other from not being the dictator, here he or she has no influence on the outcome, therefore there is no incentive to lie either.

A symmetric SD mechanism satisfies two extra conditions:

1. *Symmetric over the items*: if $\boldsymbol{v} = (v_1, \dots, v_m) \in D_i$, then $(v_{\sigma(1)}, \dots, v_{\sigma(m)}) \in D_i$, where σ is any permutation.
2. *Symmetric over the agents*: if agent i chooses $\boldsymbol{u} \in D_i$ to maximize his utility for some valuation vectors, then $\boldsymbol{u} \in D_{-i}$, where $-i$ stands for the other agent. So if we remove vectors in D_i that are never chosen by agent i, then D_1 and D_2 becomes the same. Since we only care about vectors chosen by an agent for some valuation, in the following we just use the fact that $D_1 = D_2$ for a symmetric SD mechanism.

We have the following characterization for symmetric SD mechanisms.

Theorem 1. *A symmetric strategy-proof mechanism M is SD if and only if for any valuation vector $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\alpha}$,*

$$M(\boldsymbol{u}, \boldsymbol{v}) = M(\boldsymbol{u}, \boldsymbol{\alpha}) + M(\boldsymbol{\alpha}, \boldsymbol{v}) - \frac{\mathbf{1}}{\mathbf{2}} \tag{1}$$

Proof. "$\Leftarrow$": let M be a strategy-proof symmetric mechanism that satisfies (1), then together the symmetry of the mechanism, $M(\boldsymbol{u}, \boldsymbol{v}) = M(\boldsymbol{u}, \boldsymbol{\alpha}) + M(\boldsymbol{\alpha}, \boldsymbol{v}) - \frac{\mathbf{1}}{\mathbf{2}} = M(\boldsymbol{u}, \boldsymbol{\alpha}) + (\mathbf{1} - M(\boldsymbol{v}, \boldsymbol{\alpha})) - \frac{\mathbf{1}}{\mathbf{2}} = (2M(\boldsymbol{u}, \boldsymbol{\alpha}) + \mathbf{1} - 2M(\boldsymbol{v}, \boldsymbol{\alpha}))/2$.

Note that for any fixed $\boldsymbol{\alpha}$, $\{2M(\boldsymbol{u}, \boldsymbol{\alpha}) : \boldsymbol{u}$ is a valuation vector$\}$ is very much like a choice space for the dictator, except that its component may exceed 1. To solve the problem we'll add some shift vector.

Let $\boldsymbol{c} \in \mathbb{R}^m$ be a vector such that $c_i = \min_{\boldsymbol{u}} M_i(\boldsymbol{u}, \boldsymbol{\alpha})$, for $i = 1, \dots, m$, where M_i denotes the i-th component of M and $\boldsymbol{u}$ is a valuation vector. For any valuation vectors $\boldsymbol{u}, \boldsymbol{v}$, since $\mathbf{0} \leq M(\boldsymbol{u}, \boldsymbol{v}) = M(\boldsymbol{u}, \boldsymbol{\alpha}) + M(\boldsymbol{\alpha}, \boldsymbol{v}) - \frac{\mathbf{1}}{\mathbf{2}} \leq \mathbf{1}$, and by symmetry $M(\boldsymbol{v}, \boldsymbol{\alpha}) = \mathbf{1} - M(\boldsymbol{\alpha}, \boldsymbol{v})$, we have $-1 \leq 2M_i(\boldsymbol{u}, \boldsymbol{\alpha}) - 2M_i(\boldsymbol{v}, \boldsymbol{\alpha}) \leq 1$, then $0 \leq 2M_i(\boldsymbol{u}, \boldsymbol{\alpha}) - 2c_i \leq 1$, for all $i = 1, \dots, m$.

So let

$$\{2M(\boldsymbol{u}, \boldsymbol{\alpha}) - 2\boldsymbol{c} : \boldsymbol{u} \text{ is a valuation vector}\}$$

be the choice space of the dictator. And the strategy-proof condition says $2M(\boldsymbol{u}, \boldsymbol{\alpha}) - 2\boldsymbol{c}$ maximizes $\boldsymbol{u}$'s utility from the dictator space.

"$\Rightarrow$": this direction can be easily verified from definition. □

Based on this theorem it is not hard to derive the following corollary.

Corollary 1. *A symmetric strategy-proof mechanism M is SD if and only if for any valuation vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{u}, \boldsymbol{v}$,*

$$M(\boldsymbol{u}, \boldsymbol{\alpha}) - M(\boldsymbol{v}, \boldsymbol{\alpha}) = M(\boldsymbol{u}, \boldsymbol{\beta}) - M(\boldsymbol{v}, \boldsymbol{\beta}) \tag{2}$$

3 An Upper Bound for Multiple Agents

Theorem 2. *Fix the number of items m. Let $\epsilon > 0$. There is no $(1/m + \epsilon)$-competitive strategy-proof mechanisms, for some large enough n.*

Proof. We prove by contradiction. Let $n > m$. By Proposition 2, we only need to consider symmetric mechanisms. So assume there is a symmetric mechanism M that is α-competitive on any number of agents, where $\alpha > 1/m$.

Consider the following n by m valuation matrix

$$V = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots \\ 0 & \dots & 1 & 0 & 0 \\ \frac{\epsilon}{n} & \frac{\epsilon}{n} & \dots & \frac{\epsilon}{n} & 1-(m-1)*\frac{\epsilon}{n} \\ \frac{\epsilon}{n} & \frac{\epsilon}{n} & \dots & \frac{\epsilon}{n} & 1-(m-1)*\frac{\epsilon}{n} \\ \dots \end{pmatrix}$$

For row 1 to $m-2$, each has a 1 in diagonal. Row $m-1$ to n are the same. ϵ is $2(m-1)/(\alpha m-1)$. So it's positive since $\alpha > 1/m$. By Proposition 1, for agent i where $i > m-2$, their allocation vectors are the same. Thus for each item, an agent gets at most $1/(n-m+2)$ fraction. And the overall valuations of an agent is 1. This implies the welfare of agent $(m-1)$ is bounded by $1/(n-m+2)$.

Now replace agent $(m-1)$'s type vector by $\boldsymbol{u} = (0, \dots, 0, 1, 0)$, where the $(m-1)$-th component is 1. Let V' be the changed valuation matrix.

Let $O = A(V')$. Again by Proposition 1, $o_{11} = \dots = o_{(m-1)(m-1)}$, denote it x. This is obtained by first exchange row i, j, then column i, j.

Consider the ratio under V'. The observation is, to achieve a good ratio, a mechanism will allocate a relative portion of items to the diagonal 1's. When the amount is large enough then agent $(m-1)$ has an incentive to lie from $\boldsymbol{v}_{m-1}$ to $\boldsymbol{u}$.

For item 1 to $(m-1)$, the maximal utility M can achieve is $x + (1-x)\cdot\epsilon/n$. For item m, the maximal utility M can achieve is 1. And, the optimal allocation gives a utility of $m - (m-1)\cdot\epsilon/n$. Thus we have:

$$\frac{(x + (1-x)\cdot\frac{\epsilon}{n})\cdot(m-1)+1}{m-(m-1)\cdot\frac{\epsilon}{n}} \geq \alpha \Rightarrow x \geq \frac{\frac{\alpha(m-(m-1)\cdot\frac{\epsilon}{n})-1}{m-1} - \frac{\epsilon}{n}}{1-\frac{\epsilon}{n}} \tag{3}$$

Under allocation O, the welfare of agent $(m-1)$ in valuation matrix V' is at least $x\epsilon/n$. Meanwhile, in (3), the rightmost formula has limit $(\alpha m-1)/(m-1)$, as n grows to infinity. Recall $\epsilon = 2(m-1)/(\alpha m-1)$, so formula $x\cdot\epsilon\cdot(n-m+2)/n$ has limit at least 2 as n becomes infinite. This implies, in particular, for some some large enough n, we have $x\epsilon/n > 1/(n-m+2)$. So here when agent $(m-1)$ honestly reports his valuation vector in V, the maximal welfare that can be achieved is $1/(n-m+2)$. But when he or she lies as $(0,\dots,0,1,0)$, the welfare is greater than $x\epsilon/n > 1/(n-m+2)$, which contradicts with that M is strategy-proof. □

Note that a $1/m$-competitive mechanism trivially exists: just consider the mechanism that evenly divides each item to each agent. So efficiency really becomes an issue here when there are too many people.

By a refined analysis of the above proof, we can obtain another result quite different in taste.

Theorem 3. *There does not exist a strategy-proof mechanism that achieves a competitive ratio better than $4/\sqrt{n}$, when $m \geq \sqrt{n}$.*

We leave the proof in the Appendix. This theorem also implies: as the number of agents and items approaches infinite, the competitive ratio of any strategy-proof mechanism approaches 0.

4 Allocation between Two Agents

4.1 An Upper Bound for Swap-dictatorial Mechanisms

Now we come to the setting of two agents. As mentioned above, SD is very intuitive, so it becomes very helpful for designing strategy-proof mechanisms. However, the ratio of SD may not be very good, as shown by the following theorem:

Theorem 4. *The competitive ratio of any swap-dictatorial mechanism is less than $1/2 + 1/\sqrt{\lceil \log m \rceil}$.*

Proof. Again by Proposition 2 it suffices to consider symmetric mechanisms. Let M be a symmetric SD mechanism with a competitive ratio of $1/2+\delta$. Let O be dictator's choice space. Let $m_1 = 2^{\lceil \log m \rceil}$, $m_{i+1} = m_i/2$.

We define a series of variables for case i. First let the two agent's valuation vectors be:

$$\boldsymbol{u}_i = (x,\dots,x,y,\dots,y,0,\dots,0)$$
$$\boldsymbol{v}_i = (y,\dots,y,x,\dots,x,0,\dots,0)$$

where there are $m_i/2$ consecutive x, y respectively and $y/x = t = \delta < 1$. And when agent 1 acts as the dictator, he or she chooses vector $\boldsymbol{o}_i \in O$. Vector $\boldsymbol{o}_i$ is associated with two parameters, a_i, b_i, indicating the average allocation on the portions of x, y respectively. We will show that as i increases, a_i increases by a relative amount in order to keep up the competitive ratio. However a_i cannot be greater than 1, from this seemingly contradiction we derive a bound on the competitive ratio.

By Proposition 1, when agent 2 becomes dictator, it picks $\boldsymbol{o}_i \in O$ with some permutation. So it also takes on average a_i of the x part and b_i of the y part.

Now we compute the ratio for such an allocation, by definition it is greater than $\frac{1}{2} + \delta$:

$$(x \cdot a_i + y \cdot b_i + y \cdot (1 - a_i) + x \cdot (1 - b_i)) \cdot \frac{m_i}{2} \cdot \frac{1}{2} \cdot 2 \geq (\frac{1}{2} + \delta) \cdot x \cdot m_i$$

Note that the optimal utility comes from allocating the first $m_i/2$ items to agent 1 and the next $m_i/2$ items to agent 2.

Rearrange the inequality, we get

$$a_i - b_i \geq \frac{2\delta - t}{1 - t} \tag{4}$$

On the other hand, since agent 1 chooses $\boldsymbol{o}_i$ from the dictator space to maximize utility, it must be greater than that obtained from choosing $\boldsymbol{o}_{i-1}$, as has been obtained from case $i - 1$. And, by symmetry there is a permutation of $\boldsymbol{o}_{i-1}$ in O such that the average of the first $m_i/2$ components is no less than the average of the second $m_i/2$ components. Denote it $\boldsymbol{o}$. By comparing agent 1's utility between choosing $\boldsymbol{o}_i$ and $\boldsymbol{o}$, we obtain:

$$(x \cdot a_i + y \cdot b_i) \cdot m_i/2 \geq (x + y) \cdot a_{i-1} \cdot m_i/2$$
$$\Rightarrow a_i \frac{1}{t+1} + b_i \frac{t}{t+1} \geq a_{i-1}$$

Together with (4) we get $a_i \geq a_{i-1} + t(2\delta - t)/(1 - t^2)$. Since $a_1 \geq 0$, we obtain $a_k \geq (k - 1) \cdot t(2\delta - t)/(1 - t^2)$. Let $k = [\log m]$, then:

$$([\log m] - 1) \frac{t(2\delta - t)}{1 - t^2} \leq a_k \leq 1$$

Substitute t by δ, we have $\delta \leq 1/\sqrt{[\log m]}$. □

4.2 Relation between Swap-dictatorial and Strategy-proof Mechanisms

The family of SD mechanism is one kind of strategy-proof mechanisms in our model. Together with symmetry it becomes a useful tool for designing strategy-proof mechanisms. However, it is the only family of strategy-proof mechanism we have found yet, except under some variations like letting the non-dictator choose from a set of allocations that all maximize the utility of the dictator, So, could there be any relation between these concepts? In this subsection we will give a partial result. Before discussing this question, we need to introduce some notations first.

Let $M : \mathbf{V}^2 \to [0,1]^m$ be a mechanism. Let $\boldsymbol{u}$, $\boldsymbol{v}$ be two valuation vectors. Define

$$F : S^2 \to [0,1]^m \tag{5}$$

where $S = \{(x_1, \ldots, x_{m-1}) : 0 \leq \sum_{i=1}^{m-1} x_i \leq 1 \text{ and } x_i \geq 0,\ \forall\ 1 \leq i \leq m-1\}$. And $F(u_1, \ldots, u_{m-1}, v_1, \ldots, v_{m-1}) = M(\boldsymbol{u}, \boldsymbol{v})$.

Basically this definition isolates variables upon which a mechanism is defined. It is essential here since we are going to analyze a mechanism mathematically.

Let $int(S)$ be the *interior* of S, i.e., when $0 < \sum_{i=1}^{m-1} x_i < 1$. For a slight abuse of notation, we simply use F to stand for a mechanism and when we say $\boldsymbol{u}$ is a *valuation vector*, it is a $(m-1)$-dimensional vector which can be extended as an agent's valuation. Each component of F can also be viewed as a function on S, we use f_i to denote the i-th component. These notations will be used for the rest of this subsection.

Now we are ready to define continuously differentiable mechanisms.

Definition 3 (Continuously Differentiable Mechanism). *We say a mechanism M is continuously differentiable if and only if f_i is continuously differentiable(or $f_i \in \mathcal{C}$) on T^2, for $i = 1, \ldots, m$, where $T = int(S)$.* [3]

Similarly, M is second order continuously differentiable if and only if $f_i \in \mathcal{C}^2$ when the domain is restricted to T^2, for $i = 1, \ldots, m$.

Now we'll analyze a symmetric strategy-proof mechanism M, which is also second order continuously differentiable. Let F be defined as (5). First we give another condition on whether a mechanism is SD based on differentiable assumption.

Lemma 1. *If for any valuation vectors $\boldsymbol{u}$, $\boldsymbol{v} \in int(S)$,*

$$\frac{\partial^2 F}{\partial u_i \partial v_j}(\boldsymbol{u}, \boldsymbol{v}) = \mathbf{0},\ \forall 1 \leq i, j \leq m-1 \tag{6}$$

Then M is swap-dictatorial.

Proof. For any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in int(S)$, we show that (2) can be inferred from (6). We prove the equality for the first component of F, while others follow similarly. We first do an integration to change the first parameter from $\boldsymbol{u}$ to $\boldsymbol{v}$, followed by another integration on the second parameter to change $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$.

$$(f_1(\boldsymbol{u}, \boldsymbol{\alpha}) - f_1(\boldsymbol{v}, \boldsymbol{\alpha})) - (f_1(\boldsymbol{u}, \boldsymbol{\beta}) - f_1(\boldsymbol{v}, \boldsymbol{\beta})) \tag{7}$$

$$=(f_1(\boldsymbol{u}, \boldsymbol{\alpha}) - f_1(\boldsymbol{u}, \boldsymbol{\beta})) - (f_1(\boldsymbol{v}, \boldsymbol{\alpha}) - f_1(\boldsymbol{v}, \boldsymbol{\beta})) \tag{8}$$

$$= \int_0^1 dy\,(\boldsymbol{\beta} - \boldsymbol{\alpha}) \cdot \int_0^1 dx\,(\boldsymbol{u} - \boldsymbol{v}) \nabla_x \nabla_y f_1((\boldsymbol{u} - \boldsymbol{v})x + \boldsymbol{v}, (\boldsymbol{\beta} - \boldsymbol{\alpha})y + \boldsymbol{\alpha}) \tag{9}$$

$$=0 \tag{10}$$

[3] We take the trouble to distinguish S from the interior of S, since a continuously differentiable function can only be defined on an open set.

Here $\nabla_x \nabla_y f_1$ is a $(m-1) \times (m-1)$ matrix and $(\nabla_x \nabla_y f_1)_{i,j} = \frac{\partial^2 f_1}{\partial x_i \partial y_j}(\boldsymbol{x}, \boldsymbol{y})$. Since $f_1 \in \mathcal{C}^2$, $(\boldsymbol{u}-\boldsymbol{v})\nabla_x \nabla_y f_1$ is actually $\|\boldsymbol{u}-\boldsymbol{v}\|$ times the directional derivative of $\nabla_y f_1$ along the direction of $\boldsymbol{u}-\boldsymbol{v}$.

So (7) holds in the interior of S. Since f_1 is continuous, by taking a limit it also holds in S. □

Now we are about to give the main result of this subsection. Before that, we first need Clairaut's theorem[5].

Lemma 2 (Clairaut's Theorem). *If $f : \mathbb{R}^n \to \mathbb{R}$ has continuous second partial derivatives at any given point in $\mathbb{R}^n$, say, $(a_1, a_2, \ldots, a_n)$, then for $1 \le i, j \le n$,*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(a_1, \ldots, a_n) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a_1, \ldots, a_n)$$

In words, the partial derivations of this function are commutative at that point.

Lemma 3. *For any valuation vectors $\boldsymbol{u}, \boldsymbol{v} \in int(S)$, we have:*

$$\frac{\partial^2 F}{\partial u_i \partial v_j}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{0},\ \forall\ 1 \le i, j \le m-1 \tag{11}$$

Proof. We merely prove for f_1, the first component of F, without loss of generality. By the strategy-proof condition, the first agent can't make more profits by misreporting his valuation vector, so

$$\frac{\partial f_1}{\partial u_i}(\boldsymbol{u}, \boldsymbol{v}) \cdot \boldsymbol{u}' = 0$$

where $\boldsymbol{u}'$ extends $\boldsymbol{u}$ to an m-dimensional vector, the last component being $1 - \sum_{i=1}^{m-1} u_i$. Taking a partial derivative on v_j,

$$\frac{\partial}{\partial v_j} \frac{\partial f_1}{\partial u_i}(\boldsymbol{u}, \boldsymbol{v}) \cdot \boldsymbol{u}' = 0$$

The derivative can be pushed inside the inner product because $\boldsymbol{u}$ is independent of v_j. Exchanging the role of $\boldsymbol{u}, \boldsymbol{v}$, similarly we get

$$\frac{\partial}{\partial u_i} \frac{\partial f_1}{\partial v_j}(\boldsymbol{u}, \boldsymbol{v}) \cdot \boldsymbol{v}' = 0$$

$\boldsymbol{v}'$ is defined similarly to $\boldsymbol{u}'$.

Since f_1 has continuous second partial derivative at $(\boldsymbol{u}, \boldsymbol{v})$, by Clairaut's theorem

$$\frac{\partial^2 f_1}{\partial u_i \partial v_j}(\boldsymbol{u}, \boldsymbol{v}) = \frac{\partial}{\partial v_j} \frac{\partial f_1}{\partial u_i}(\boldsymbol{u}, \boldsymbol{v}) = \frac{\partial}{\partial u_i} \frac{\partial f_1}{\partial v_j}(\boldsymbol{u}, \boldsymbol{v})$$

In conclusion, $\frac{\partial^2 f_1}{\partial u_i \partial v_j}$ is simultaneously perpendicular to $\boldsymbol{u}', \boldsymbol{v}' \in \mathbb{R}^2$. When $\boldsymbol{u} \ne \boldsymbol{v}$ (i.e. $\boldsymbol{u}' \ne \boldsymbol{v}'$) , it must be the case that $\frac{\partial^2 f_1}{\partial u_i \partial v_j}(u, v) = \boldsymbol{0}$. Since $\frac{\partial^2 f_1}{\partial u_i \partial v_j}(u, v)$ is continuous, it is $\boldsymbol{0}$ when $\boldsymbol{u}$ equals $\boldsymbol{v}$ too. □

Combining the results of Lemma 1 and Lemma 3, we have

Theorem 5. *In the case of allocating 2 items to 2 agents, if a mechanism M is symmetric and second order continuously differentiable, then M is strategy-proof if and only if M is swap-dictatorial.*

We remark that the assumption of second order continuously differentiable can be extended to the case when the function may have finitely many discontinuous points, since integration can be done in that case too. To find a function beyond this assumption, one may need to think about some quite unnatural functions.

We hope Theorem 5 helps to explain the difficulty we encountered in designing strategy-proof mechanisms that are not SD. It will be interesting to see if Theorem 5 still holds in more general cases, or if there exists other families of strategy-proof mechanisms.

4.3 Bounded Valuation

Section 4.2 gives evidence that symmetric strategy-proof allocation may be, actually, swap-dictatorial. And from Theorem 4 SD can only achieve a competitive ratio of 0.5 when the number of items approaches infinity, which is no better than an even allocation. But can SD do better than even allocation, when we impose some restrictions on the valuation vectors? In this subsection, we see that if an agent's valuation is not too biased, SD can do better than even allocation. To put it formally, we define:

Definition 4 (Bounded Valuation). *Let $\boldsymbol{v} = (v_1, \ldots, v_m)$ be a valid valuation vector, we say $\boldsymbol{v}$ is bounded by T, if $v_i \le T$, for any $i = 1, \ldots, m$. A valuation space is bounded by T if each vector of the space is bounded by T.*

Let T be C/m. Note that if we allow C to grow arbitrarily large as m grows, then a proof similar to Theorem 4's shows that there is still no SD mechanism that is $(0.5+\epsilon)$-competitive on a valuation space bounded by C/m, for any $\epsilon > 0$. However, when C is some fixed constant, the proof no longer holds, and we can actually find an SD mechanism that does better than 0.5.

Definition 5 (Sphere Mechanism). *Let $f(\boldsymbol{u}) = (\frac{u_1 \cdot c}{\|\boldsymbol{u}\|}, \frac{u_2 \cdot c}{\|\boldsymbol{u}\|}, \ldots, \frac{u_m \cdot c}{\|\boldsymbol{u}\|})$, where $c = \sqrt{m}/C$ and $\|\cdot\|$ denotes the L_2-norm. Given two valuation vectors $\boldsymbol{u}$, $\boldsymbol{v}$,*

$$M(\boldsymbol{u}, \boldsymbol{v}) = \frac{f(\boldsymbol{u}) + \mathbf{1} - f(\boldsymbol{v})}{2} \tag{12}$$

Here c is chosen appropriately so that each component of $f(\boldsymbol{u})$ is no larger than 1.

Our SD has nice mathematical interpretations. The choice space for the dictator is:

$$D = \left\{ \frac{c}{\|\boldsymbol{u}\|}\boldsymbol{u} : \boldsymbol{u} \text{ is a valuation vector bounded by } C/m \right\}$$

So all the vectors in the choice space are in a sphere of radius c. To maximize utility, the dictator will choose the vector of the same direction to its valuation vector, i.e., a dictator with valuation vector $\boldsymbol{u}$ will choose $c\boldsymbol{u}/\|\boldsymbol{u}\|$, as M does in (12). Since SD is always strategy-proof, the Sphere mechanism is strategy-proof as well.

Next we analyze the competitive ratio.

Theorem 6. *Let the valuation space $\boldsymbol{V}$ be bounded by C/m. Then Sphere mechanism is $(\frac{1}{2}+\epsilon)$-competitive, for some $\epsilon > 0$.*

Let's prove two lemmas first.

Lemma 4. *Let $\gamma(\boldsymbol{u},\boldsymbol{v})$ be the optimal social welfare. Then $\gamma(\boldsymbol{u},\boldsymbol{v}) = 1+\frac{1}{2}\sum_{i=1}^{m}|u_i - v_i|$*

Proof. $\gamma(\boldsymbol{u},\boldsymbol{v}) = \sum_{i=1}^{m} max(u_i, v_i) = \sum_{i=1}^{m}(u_i + v_i)/2 + |u_i - v_i|/2 = 1 + \sum_{i=1}^{m}|u_i - v_i|/2$ □

Lemma 5. *For any $\boldsymbol{v} \in \boldsymbol{V}$, $\|\boldsymbol{v}\| \in [1/\sqrt{m}, \sqrt{C/m}]$.*

Proof. By Cauchy-Schwarz inequality, $\|\boldsymbol{v}\| \geq 1/\sqrt{m}$. The other part comes since $\boldsymbol{v}$ is bounded by C/m. □

Then we prove Theorem 6.

Proof. Now we compute the ratio of M under $\boldsymbol{u}$, $\boldsymbol{v}$:

$$\begin{aligned}\alpha(\boldsymbol{u},\boldsymbol{v}) =& \frac{\sum_{i=1}^{m}(u_i(1+cu_i/\|\boldsymbol{u}\| - cv_i/\|\boldsymbol{v}\|) + v_i(1-cu_i/\|\boldsymbol{u}\| + cv_i/\|\boldsymbol{v}\|))}{2\gamma(\boldsymbol{u},\boldsymbol{v}))} \\ =& \frac{2 + c\|\boldsymbol{u}\| + c\|\boldsymbol{v}\| - c\boldsymbol{u}\cdot\boldsymbol{v}(1/\|\boldsymbol{u}\| + 1/\|\boldsymbol{v}\|)}{2\gamma(\boldsymbol{u},\boldsymbol{v})}\end{aligned}$$

Assume $\theta = \langle \boldsymbol{u},\boldsymbol{v}\rangle$, i.e., θ is the angle between $\boldsymbol{u},\boldsymbol{v}$, then $\boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\|\cos\theta$, and

$$\alpha(\boldsymbol{u},\boldsymbol{v}) = \frac{2 + c(\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)(1-\cos\theta)}{2\gamma(\boldsymbol{u},\boldsymbol{v})}$$

Recall that $\gamma(\boldsymbol{u},\boldsymbol{v}) = 1 + \sum_{i=1}^{m}|u_i - v_i|/2$. We now consider two cases. In the first case, $\sum_i|u_i - v_i|$ is small, and so in the denominator $\gamma(\boldsymbol{u},\boldsymbol{v})$ is small, and we get a $\alpha(\boldsymbol{u},\boldsymbol{v})$ larger than $1/2$. In the second case, $\sum_i|u_i - v_i|$ is large, but then we can prove that θ must be large, so in the numerator $(\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)(1-\cos\theta)$ is large, and we get a $\alpha(\boldsymbol{u},\boldsymbol{v})$ larger than $1/2$ as well. More formally, let $0 < \beta < 2$ be a parameter.

Case 1 $\sum_{i=1}^{m}|u_i - v_i| \leq \beta$. Then $\alpha(\boldsymbol{u},\boldsymbol{v}) > \frac{2}{2\gamma(\boldsymbol{u},\boldsymbol{v})} \geq \frac{2}{2+\beta}$.
Case 2 $\sum_{i=1}^{m}|u_i - v_i| > \beta$. Then using CauchySchwarz inequality

$$\cos\theta = \frac{\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - \|\boldsymbol{u}-\boldsymbol{v}\|^2}{2\|\boldsymbol{u}\|\|\boldsymbol{v}\|} \leq \frac{\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - \beta^2/m}{2\|\boldsymbol{u}\|\|\boldsymbol{v}\|}$$

Since $1/\sqrt{m} \le \|\boldsymbol{u}\|, \|\boldsymbol{v}\| \le \sqrt{C/m}$, we get that $\cos\theta < g(C,\beta) < 1$ for some function g. And so

$$\alpha(\boldsymbol{u},\boldsymbol{v}) = \frac{2 + \frac{\sqrt{m}}{C}(\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)(1-\cos\theta)}{2\gamma(\boldsymbol{u},\boldsymbol{v})} \ge \frac{1 + (1 - g(C,\beta))/C}{2}$$

Combining the two cases, we see that for any β,

$$\alpha(\boldsymbol{u},\boldsymbol{v}) \ge \min(\frac{2}{2+\beta}, \frac{1 + (1 - g(C,\beta))/C}{2}) > \frac{1}{2}$$

□

5 Conclusions and Future Research

In this paper we studied resource allocation problem when there are no payments or priors. While this model is only proposed recently, we hope the results and proof techniques in this paper provide insight into the model. There are still several problem unsettled for this problem. The first is whether there exists a strategy-proof mechanism that beats the 0.5 ratio. There's still a large gap here since the only known result is a 0.841 upper bound. The second is to what extent are strategy-proof equivalent to swap-dictatorial mechanisms in this model. Another direction for future research would be to consider other social optimality index like egalitarian criterion, or handle issues like fairness in the model.

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A Proof of Theorem 3

Proof. By Proposition 2, we only need to consider symmetric mechanisms. Let M be any symmetric mechanism where $m \geq \sqrt{n}$. Let α be its competitive ratio. Let $k = [\sqrt{n}]$.

Consider the following valuation matrix.

$$V = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots \\ 0 & 1 & \dots & 0 & 0 & \dots \\ \dots \\ 0 & \dots & 1 & 0 & 0 & \dots \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \dots & \frac{1}{\sqrt{n}} & 1-\frac{k-1}{\sqrt{n}} & \dots \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \dots & \frac{1}{\sqrt{n}} & 1-\frac{k-1}{\sqrt{n}} & \dots \\ \dots \end{pmatrix}$$

Column k+1 to m are all zero. $m \geq k$ is nonnegative, since $m \geq \sqrt{n}$. For row 1 to $k-2$, each has a 1 in diagonal. Row $k-1$ to n are the same. This matrix is essentially the same as in the proof of Theorem 2, if we let $\epsilon = \sqrt{n}$ there. By Proposition 1, for agent i where $i \geq k-1$, their allocation vectors are the same. This implies agent (k-1) has a maximal welfare of $1/(n-k+2)$.

Now replace agent (k - 1)'s type vector by $\boldsymbol{u} = (0, \dots, 0, 1, 0, \dots)$, where the (k-1)-th component is 1. Let V' be the changed valuation matrix.

Let $O = M(V')$. Again by Proposition 1, $o_{11} = \cdots = o_{(m-1)(m-1)}$, denote it x. This is obtained by first exchange row i, j, then column i, j.

Consider the ratio under V'. We have:

$$\frac{(x + \frac{1-x}{\sqrt{n}}) \times (k-1) + 1}{k - \frac{k-1}{\sqrt{n}}} \geq \alpha \tag{13}$$

$$\Rightarrow x \geq \frac{\frac{\alpha(k-(k-1)/\sqrt{n})-1}{k-1} - \frac{1}{\sqrt{n}}}{1 - \frac{1}{\sqrt{n}}} \tag{14}$$

Under allocation O, the utility of agent $(m-1)$ in valuation matrix V' is at least $x/\sqrt{n}$. On the other hand, it must be smaller than $1/(n-k+2)$. Otherwise in V agent $(k-1)$ has the incentive to misreport his type vector as $\boldsymbol{u}$, which contradicts with that A is strategy-proof.

As a result, $1/(n-k+2) \geq x/\sqrt{n}$. Combined with (14) we have an upper bound for α. Then it's easy to get the desired bound. ◻

B The ratio of Sphere mechanism

To get a better idea of what competitive ratios we are talking about in Theorem 6, we may choose appropriate values of β to achieve good competitive ratios under different C. Some of the numerical results are summarized in Table 1.

C	β	α
2	1.03	0.658
3	1.2	0.624
10	1.6	0.545

Table 1. For different C, the values of β we choose and the corresponding competitive ratios α

.